ISRAEL JOURNAL OF MATHEMATICS **164** (2008), 153–164
DOI: 10.1007/s11856-008-0024-3

FACE VECTORS OF FLAG COMPLEXES

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ABSTRACT

A conjecture of Kalai and Eckhoff that the face vector of an arbitrary flag complex is also the face vector of some particular balanced complex is verified.

1. Introduction

We begin by introducing the main result. Precise definitions and statements of some related theorems are deferred to later sections.

The main object of our study is the class of flag complexes. A simplicial complex is a **flag complex** if all of its minimal non-faces are two element sets. Equivalently, if all of the edges of a potential face of a flag complex are in the complex, then that face must also be in the complex.

Flag complexes are closely related to graphs. Given a graph G, define its **clique complex** $C = C(G)$ as the simplicial complex whose vertex set is the vertex set of G, and whose faces are the cliques of G. The clique complex of any graph is itself a flag complex, as for a subset of vertices of a graph to not form a clique, two of them must not form an edge. Conversely, any flag complex is the clique complex of its 1-skeleton.

The Kruskal–Katona theorem [6, 5] characterizes the face vectors of simplicial complexes as being precisely the integer vectors whose coordinates satisfy some particular bounds. The graphs of the "rev-lex" complexes which attain these

Received June 04, 2006

bounds invariably have a clique on all but one of the vertices of the complex, and sometimes even on all of the vertices.

Since the bounds of the Kruskal–Katona theorem hold for all simplicial complexes, they must in particular hold for flag complexes. We might expect that flag complexes which do not have a face on most of the vertices of the complex will not come that close to attaining the bounds of the Kruskal–Katona theorem.

One way to force tighter bounds on face numbers is by requiring the graph of the complex to have a chromatic number much smaller than the number of vertices. The face vectors of simplicial complexes of a given chromatic number were classified by Frankl, Füredi, and Kalai [4].

Kalai (unpublished; see [8, p. 100]) and Eckhoff [1] independently conjectured that if the largest face of a flag complex contains r vertices, then it must satisfy the known bounds (see [4]) for complexes of chromatic number r, even though the flag complex may have chromatic number much larger than r. We prove their conjecture.

Theorem 1.1: *For any flag complex C, there is a balanced complex C' with the same face vector as C.*

Our proof is constructive. The Frankl–Füredi–Kalai [4] theorem states that an integer vector is the face vector of a balanced complex if and only if it is the face vector of a colored "rev-lex" complex. This happens if and only if it satisfies certain bounds on consecutive face numbers. Given a flag complex, for each i, we construct a colored "rev-lex" complex with the same number of i-faces and $(i+1)$-faces as the flag complex, thus showing that all the bounds are satisfied.

The structure of the paper is as follows. Section 2 contains basic facts and definitions related to simplicial complexes. In Section 3, we discuss the Kruskal–Katona theorem and the Frankl–Füredi–Kalai theorem, and lay the foundation for our proof. Finally, Section 4 gives our proof of the Kalai–Eckhoff conjecture.

2. Preliminaries on simplicial complexes

In this section, we discuss some basic definitions related to simplicial complexes.

Recall that a **simplicial complex** Δ on a vertex set V is a collection of subsets of V such that, (i) for every $v \in V$, $\{v\} \in \Delta$ and (ii) for every $B \in \Delta$, if $A \subset B$, then $A \in \Delta$. The elements of Δ are called **faces**. The maximal faces (under inclusion) are called **facets**.

For a face F of a simpicial complex Δ, the **dimension** of F is defined as $\dim F = |F| - 1$. The dimension of Δ, $\dim \Delta$, is defined as the maximum dimension of the faces of Δ. A complex Δ is **pure** if all of its facets are of the same dimension.

The i-skeleton of a simplicial complex Δ is the collection of all faces of Δ of dimension $\leq i$. In particular, the 1-skeleton of Δ is its underlying graph.

It is sometimes useful in inductive proofs to consider certain subcomplexes of a given simplicial complex, such as its links.

Definition 2.1: Let Δ be a simplicial complex and $F \in \Delta$. The **link** of F, $\mathrm{lk}_\Delta(F)$, is defined as

$$\mathrm{lk}_\Delta(F) := \{G \in \Delta\colon F \cap G = \emptyset, F \cup G \in \Delta\}.$$

The link of a face of a simplicial complex is itself a simplicial complex. It will be convenient to define the notion of a link of a vertex of a graph.

Definition 2.2: The link of a vertex v in a graph G, denoted $\mathrm{lk}_G(v)$, is the induced subgraph of G on all vertices adjacent to v.

Note that $\mathrm{lk}_G(v)$ coincides with the 1-skeleton of the link of $\{v\}$ in the clique complex of G.

Next we discuss a special class of simplical complexes known as flag complexes.

Definition 2.3: A simplicial complex Δ on a vertex set V is a **flag complex** if all of its minimal non-faces are two element sets. A non-face of Δ is a subset $A \subseteq V$ such that $A \notin \Delta$. A non-face A is minimal if, for all proper subsets $B \subset A$, $B \in \Delta$.

In the following, we refer to the chromatic number of a simplicial complex as the chromatic number of its 1-skeleton in the usual graph theoretic sense.

We also need the notion of a balanced complex, as introduced and studied in [7].

Definition 2.4: A simplicial complex Δ of dimension $d-1$ is **balanced** if it has chromatic number d.

Note that the chromatic number of a simpicial complex of dimension $d-1$ must be at least d, as it has some face with d vertices, all of which are adjacent, so coloring that face takes d colors. A balanced complex is then one whose chromatic number is no larger than it has to be.

Not all simplicial complexes are balanced complexes. For example, a pentagon (five vertices, five edges, and one empty face) is not a balanced complex, because it has chromatic number three but dimension only one.

In this paper, we study the face numbers of flag complexes.

Definition 2.5: The **i-th face number** of a simplicial complex C, denoted $c_i(C)$, is the number of faces in C containing i vertices. These are also called **i-faces** of C. If $\dim C = d-1$, the **face vector** of C is the vector X

$$c(C) = (c_0(C), c_1(C), \ldots, c_d(C)).$$

In particular, for any non-empty complex C, we have $c_0(C) = 1$, as there is a unique empty set of vertices, and it is a face of C.

Since flag complexes are the same as clique complexes of graphs, it is sometimes convenient to talk about face numbers in the language of graphs.

Definition 2.6: The **i-th face number** of a graph is the i-th face number of its clique complex. Likewise, the **clique vector** of a graph is the face vector of its clique complex.

The face numbers defined here are shifted by one from what is often used for simplicial complexes. This is done because we are primarily concerned with flag complexes, or equivalently, clique complexes of graphs, where it is more natural to index i as the number of vertices in a clique of the graph, following Eckhoff [3].

The graph concept corresponding to the dimension of a simplicial complex is the clique number.

Definition 2.7: The **clique number** of a graph is the number of vertices in its largest clique.

Note that the clique number of a graph is one larger than the dimension of its clique complex.

3. The Kruskal-Katona and Frankl–Füredi-Kalai theorems

For the general case of simplicial complexes, the question of which face vectors are possible is answered by the Kruskal–Katona theorem [6, 5]. Stating the theorem requires the following lemma.

Lemma 3.1: *Given any positive integers m and k, there is a unique s and unique $n_k > n_{k-1} > \cdots > n_{k-s} \geq k-s > 0$ such that*

$$m = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_{k-s}}{k-s}.$$

The representation described in the lemma is called the k-canonical representation of m.

Theorem 3.2 (Kruskal–Katona): *For a simplicial complex C, let*

$$m = c_k(C) = \binom{n_k}{k} + \binom{n_{k-1}}{k-1} + \cdots + \binom{n_{k-s}}{k-s}$$

be the k-canonical representation of m. Then

$$c_{k+1}(C) \leq \binom{n_k}{k+1} + \binom{n_{k-1}}{k} + \cdots + \binom{n_{k-s}}{k-s+1}.$$

Furthermore, given a vector $(1, c_1, c_2, \ldots, c_t)$ which satisfies this bound for all $1 \leq k < t$, there is some complex that has this vector as its face vector.

To construct the complexes which demonstrate that the bound of the Kruskal-Katona theorem is attained, we need the reverse-lexicographic ("rev-lex") order. To define the rev-lex order of i-faces of a simplicial complex on n vertices, we start by labelling the vertices $1, 2, \ldots$. Let $\mathbb{N}$ be the natural numbers, let A and B be distinct subsets of $\mathbb{N}$ with $|A| = |B| = i$, and let $A \nabla B$ be the symmetric difference of A and B.

Definition 3.3: For $A, B \subset \mathbb{N}$ with $|A| = |B|$, we say that A precedes B in the rev-lex order if $\max(A \nabla B) \in B$, and B precedes A otherwise.

For example, $\{2, 3, 5\}$ precedes $\{1, 4, 5\}$, as 3 is less than 4, and $\{3, 4, 5\}$ precedes $\{1, 2, 6\}$.

Definition 3.4: The **rev-lex complex on m i-faces** is the pure complex whose facets are the first m i-sets possible in rev-lex order. This complex is denoted $C_i(m)$.

We can also specify more than one number in the face vector. For two sequences $i_1 < \cdots < i_r$ and $(m_1, \ldots, m_r)$, let

$$C = C_{i_1}(m_1) \cup C_{i_2}(m_2) \cup \cdots \cup C_{i_r}(m_r).$$

A standard way to prove the Kruskal–Katona theorem involves showing that if the numbers $m_1, \ldots, m_r$ satisfy the bounds of the theorem, then the complex C has exactly m_j i_j-faces for all $j \leq r$ and no more. In this case, we refer to C as the rev-lex complex on m_1 i_1-faces, $\ldots$, m_r i_r-faces.

For example, if the complex C has $\binom{9}{3} + \binom{6}{2} = 99$ 3-faces, then the Kruskal-Katona theorem says that it can have at most $\binom{9}{4} + \binom{6}{3} = 146$ 4-faces. The rev-lex complex on 99 3-faces and 146 4-faces gives an example showing that this bound is attained.

The 1-skeleton of the rev-lex complex that gives the example for the existence part of the Kruskal–Katona theorem always has a clique as large as possible without exceeding the number of edges allowed. It also has a chromatic number of either the number of non-isolated vertices or one less than this, as the edges form a clique on all of the non-isolated vertices except possibly for the last one. It turns out that if we require a much smaller chromatic number, we can get a much smaller bound. To take an extreme example, if $c_3(C) = 1140$, then the Kruskal–Katona theorem requires that $c_4(C) \leq 4845$. But if we require the complex C to be 3-colorable, then we trivially cannot have any faces on four vertices, and $c_4(C) = 0$.

We could ask what face vectors occur for r-colorable complexes for a given r. This was solved by Frankl, Füredi, and Kalai [4]. In order to explain their result, we need the concept of a Turán graph.

Definition 3.5: The Turán graph $T_{n,r}$ is the graph obtained by partitioning n vertices into r parts as evenly as possible, and making two vertices adjacent exactly if they are not in the same part. Define $\binom{n}{k}_r$ to be the number of k-cliques of the graph $T_{n,r}$.

The structure of the Frankl–Füredi–Kalai theorem [4] is similar to that of the Kruskal–Katona theorem, beginning with a canonical representation of the number of faces.

LEMMA 3.6: *Given positive integers m, k, and r with $r \geq k$, there are unique $s, n_k, n_{k-1}, \ldots, n_{k-s}$ such that*

$$m = \binom{n_k}{k}_r + \binom{n_{k-1}}{k-1}_{r-1} + \cdots + \binom{n_{k-s}}{k-s}_{r-s},$$

$n_{k-i} - \left\lfloor \frac{n_{k-i}}{r-i} \right\rfloor > n_{k-i-1}$ *for all* $0 \leq i < s$, *and* $n_{k-s} \geq k-s > 0$.

This expression is called the (k,r)-canonical representation of m.

THEOREM 3.7 (Frankl–Füredi–Kalai): *For an r-colorable complex C, let*

$$m = c_k(C) = \binom{n_k}{k}_r + \binom{n_{k-1}}{k-1}_{r-1} + \cdots + \binom{n_{k-s}}{k-s}_{r-s}$$

be the (k,r)-canonical representation of m. Then

$$c_{k+1}(C) \leq \binom{n_k}{k+1}_r + \binom{n_{k-1}}{k}_{r-1} + \cdots + \binom{n_{k-s}}{k-s+1}_{r-s}.$$

Furthermore, given a vector $(1, c_1, c_2, \ldots c_t)$ which satisfies this bound for all $1 \leq k < t$, there is some r-colorable complex that has this vector as its face vector.

The examples which show that this bound is sharp come from a colored equivalent of the rev-lex complexes of the Kruskal–Katona theorem:

Definition 3.8: A subset $A \subset \mathbb{N}$ is r-**permissible** if, for any two $a, b \in A$, r does not divide $a - b$. The r-**colored rev-lex complex on** m i-**faces** is the pure complex whose facets are the first m r-permissible i-sets in rev-lex order. This complex is denoted $C_i^r(m)$.

The complex $C_i^r(n)$ is r-colorable because we can color all vertices which are i modulo r with color i.

As with the uncolored case, we can define a rev-lex complex with specified face numbers of more than one dimension. For two sequences $i_1 < \cdots < i_s$ and $(m_1, \ldots, m_s)$, let $C = C_{i_1}^r(m_1) \cup C_{i_2}^r(m_2) \cup \cdots \cup C_{i_s}^r(m_s)$. The proof of Theorem 3.7 involves showing that if the numbers $m_1, \ldots, m_r$ satisfy the bounds of the theorem, then the complex C has exactly m_j i_j-faces and no more. In this case, we refer to C as the r-colored rev-lex complex on m_1 i_1-faces, $\ldots$, m_r i_r-faces. This complex is likewise r-colorable with one color for each value modulo r.

In the case of flag complexes, the face numbers of the complex must still follow the bounds imposed by the chromatic number by Theorem 3.7. Still, there are graphs whose clique number is far smaller than the chromatic number, and having no large cliques seems to force tighter restrictions on the clique vector than the chromatic number alone. In particular, given a graph G of clique number n, we must have $c_i(G) = 0$ for all $i > n$, while the bound from the chromatic number and Theorem 3.7 may be rather large. Note that the chromatic number must be at least the size of the largest clique, as any two vertices in a maximum size clique must have different colors.

It has been conjectured by Kalai (unpublished) and Eckhoff [1] that, given a graph G with clique number r, there is an r-colorable complex with exactly the same face numbers as the clique complex of the graph. Their conjecture generalizes the classical Turán theorem from graph theory, which states that among all triangle-free graphs on n vertices, the Turán graph $T_{n,2}$ has the most edges [9]. The goal of the following section is to verify Theorem 1.1, proving their conjecture.

The reverse inclusion does not hold. For example, the vector $(1, 4, 5, 1)$ in the case of $r = 3$ satisfies the bounds of the Frankl–Füredi–Kalai theorem, so it is the face vector of a balanced complex. However, it is not the face vector of a flag complex: a graph with four vertices and five edges must be one edge short of a clique on four vertices, which has two triangles, not one.

4. Proof of the Kalai-Eckhoff conjecture

Fix a graph G with $c_{r+1}(G) = 0$ and fix $k \geq 0$. We start by showing that there is an r-colorable complex C with $c_k(G) = c_k(C)$ and $c_{k+1}(G) = c_{k+1}(C)$ (see Lemma 4.1 below).

The case $k = 1$ of the lemma is given by Turán's theorem [9]. It was generalized by Zykov [10] to state that if G is a graph on n vertices of chromatic number r, then $c_i(G) \leq \binom{n}{i}_r$. The case $k = 2$ was proven by Eckhoff [2]. A subsequent paper of Eckhoff [3] established a bound on $c_i(G)$ in terms of $c_2(G)$ for all $2 \leq i$. All of these results are special cases of our Theorem 4.2 and proven independently below.

LEMMA 4.1: *If G is a graph with $c_{r+1}(G) = 0$ and k is a nonnegative integer, then there is some r-colorable complex C with $c_k(C) = c_k(G)$ and $c_{k+1}(C) = c_{k+1}(G)$.*

Proof. We use induction on k. For the base case, if $k = 0$, take C to be a complex with the same number of vertices as G and no edges, so that all vertices can be the same color.

Otherwise, assume that the lemma holds for $k - 1$, and we need to prove it for k. The approach for this is to use induction on $c_{k+1}(G)$. For the base case, if $c_{k+1}(G) = 0$, then take C to be a disjoint union of $c_k(G)$ k-faces.

For the inductive step, suppose that $c_{k+1}(G) > 0$. Let v_0 be the vertex of G contained in the most cliques of $k + 1$ vertices; in case of a tie, arbitrarily pick some vertex tied for the most to label v_0. Let the vertices of G not adjacent to v_0 be $v_1, v_2, \ldots v_s$.

Given a graph G and a vertex v, there is a bijection between k-cliques of $\mathrm{lk}_G(v)$ and $(k+1)$-cliques of G containing v, where a k-clique of $\mathrm{lk}_G(v)$ corresponds to the $(k+1)$-clique of G containing the k vertices of the k-clique of $\mathrm{lk}_G(v)$ together with v. Then the number of $(k+1)$-cliques of G containing v is $c_k(\mathrm{lk}_G(v))$. In particular, the choice of v_0 gives $c_k(\mathrm{lk}_G(v_0)) \geq c_k(\mathrm{lk}_G(v'))$ for every vertex $v' \in G$.

Define graphs $G_0, G_1, \ldots, G_{s+1}$ by setting $G_{i+1} = G - \{v_0, v_1, \ldots v_i\}$ for $0 \leq i \leq s$ and $G_0 = G$. Clearly, $G = G_0 \supset G_1 \supset \cdots \supset G_{s+1}$. Further, G_{s+1} is the induced subgraph on the vertices adjacent to v_0, which is $\mathrm{lk}_G(v_0)$.

Since $c_{r+1}(G) = 0$, $c_r(\mathrm{lk}_G(v_0)) = 0$, for, otherwise, the r vertices of an r-clique of $\mathrm{lk}_G(v_0)$ together with v_0 would form an $(r+1)$-clique of G. Then $c_r(G_{s+1}) = 0$. Further, since $c_{k+1}(G) > 0$, and v_0 is contained in the most $(k+1)$-cliques of any vertex of G, v_0 is contained in at least one $(k+1)$-clique, and so $c_k(\mathrm{lk}_G(v_0)) > 0$. Since v is contained in at least one $(k+1)$-clique of G, we have $c_{k+1}(G_{s+1}) < c_{k+1}(G)$.

Then by the second inductive hypothesis, there is some $(r-1)$-colorable complex C_{s+1} such that $c_k(C_{s+1}) = c_k(G_{s+1})$ and $c_{k+1}(C_{s+1}) = c_{k+1}(G_{s+1})$. Since given any $(r-1)$-colorable complex, there is an $(r-1)$-colorable rev-lex complex with the same face numbers, we can take C_{s+1} to be a rev-lex complex. Further, since $c_{k+1}(C_{s+1})$ and $c_k(C_{s+1})$ only force a lower bound on $c_{k-1}(C_{s+1})$, but not an upper bound, we can take $c_{k-1}(C_{s+1}) \geq c_{k-1}(G)$.

Let $c_k(\mathrm{lk}_{G_i}(v_i)) = a_i$ and $c_{k-1}(\mathrm{lk}_{G_i}(v_i)) = b_i$. Since $G_{i+1} = G_i - v_i$, $c_{k+1}(G_i) - c_{k+1}(G_{i+1}) = a_i$ and $c_k(G_i) - c_k(G_{i+1}) = b_i$. We have $c_k(\mathrm{lk}_G(v_0)) \geq c_k(\mathrm{lk}_G(v_i))$ by the choice of v_0. We also have $c_k(\mathrm{lk}_G(v_i)) \geq c_k(\mathrm{lk}_{G_i}(v_i))$ since $G_i \subset G$. Thus

$$c_k(C_{s+1}) = c_k(G_{s+1}) = c_k(\mathrm{lk}_G(v_0)) \geq c_k(\mathrm{lk}_G(v_i)) \geq c_k(\mathrm{lk}_{G_i}(v_i)) = a_i.$$

Also, since $\mathrm{lk}_{G_i}(v_i) \subset G_i \subset G$, we have

$$b_i = c_{k-1}(\mathrm{lk}_{G_i}(v_i)) \leq c_{k-1}(G_i) \leq c_{k-1}(G) \leq c_{k-1}(C_{s+1}).$$

Given an r-colored complex C_{i+1} such that $c_{k+1}(C_{i+1}) = c_{k+1}(G_{i+1})$, $c_k(C_{i+1}) = c_k(G_{i+1})$, and the induced subcomplex of C_{i+1} on the vertices of the first $r-1$ colors is isomorphic to C_{s+1}, we want to construct a complex C_i such that $c_{k+1}(C_i) = c_{k+1}(G_i)$, $c_k(C_i) = c_k(G_i)$, and the induced subcomplex of C_i on the vertices of the first $r-1$ colors is isomorphic to C_{s+1}.

Construct C_i from C_{i+1} by adding a new vertex v_i' of color r. Let the $(k+1)$-faces containing v_i' consist of each of the first a_i k-faces in rev-lex order of C_{s+1} together with v_i', and let the k-faces containing v_i' consist of each of the first b_i $(k-1)$-faces in rev-lex order of C_{s+1} together with v_i'.

If this construction can be done, then $c_{k+1}(C_i)$ is the number of $(k+1)$-faces of C_i containing v_i' plus the number of $(k+1)$-faces of C_i not containing v_i', which are a_i and $c_{k+1}(C_{i+1})$, respectively. Then

$$c_{k+1}(C_i) = c_{k+1}(C_{i+1}) + a_i = c_{k+1}(G_{i+1}) + a_i = c_{k+1}(G_i).$$

Likewise, we have

$$c_k(C_i) = c_k(C_{i+1}) + b_i = c_k(G_{i+1}) + b_i = c_k(G_i).$$

Further, it is clear from the construction that the induced subcomplex on vertices of the first $r-1$ colors is unchanged from C_{i+1}, and hence is isomorphic to C_{s+1}.

In order to show that the construction is possible, we must show that $c_k(C_{s+1}) \geq a_i$ and $c_{k-1}(C_{s+1}) \geq b_i$, and that it is possible for an $(r-1)$-colored complex C to have exactly $c_k(C) = a_i$ and $c_{k-1}(C) = b_i$. We have already shown the first two of these.

For the third, since $G_i \subset G$, we have $c_{r+1}(G_i) \leq c_{r+1}(G) = 0$, and so $c_{r+1}(G_i) = 0$. Then $c_r(\mathrm{lk}_{G_i}(v_i)) = 0$. We also have $c_k(\mathrm{lk}_{G_i}(v_i)) = a_i$ and $c_{k-1}(\mathrm{lk}_{G_i}(v_i)) = b_i$ by the definitions of a_i and b_i. Then by the first inductive hypothesis, there is some $(r-1)$-colored complex C_i' such that $c_k(C_i') = a_i$ and

$c_{k-1}(C'_i) = b_i$. Then we can take C'_i to be the $(r-1)$-colored rev-lex complex with $c_k(C'_i) = a_i$ and $c_{k-1}(C'_i) = b_i$. Since C_{s+1} is an $(r-1)$-colored rev-lex complex with $c_k(C_{s+1}) \geq a_i$ and $c_{k-1}(C_{s+1}) \geq b_i$, $C'_i \subset C_{s+1}$, and we can choose the link of v'_i in C_i to be C'_i.

We can repeat this construction for each $0 \leq i \leq s$ to start with C_{s+1}, then construct C_s, then C_{s-1}, and so forth, until we have an r-colored complex C_0 such that $c_k(C_0) = c_k(G)$ and $c_{k+1}(C_0) = c_{k+1}(G)$. This completes the inductive step for the induction on $c_{k+1}(G)$, which in turn completes the inductive step for the induction on k. ∎

We are now ready to prove the result which immediately implies Theorem 1.1, and hence establish the Kalai–Eckhoff conjecture, by taking r to be the clique number of G.

Theorem 4.2: *For every graph G with $c_{r+1}(G) = 0$, there is an r-colorable complex C such that $c_i(C) = c_i(G)$ for all i.*

Proof. By Lemma 4.1, we can pick an r-colored complex C_i such that $c_i(C_i) = c_i(G)$ and $c_{i+1}(C_i) = c_{i+1}(G)$ for all $i \geq 1$. By Theorem 3.7, we can take C_i to be the rev-lex complex on $c_i(G)$ i-faces and $c_{i+1}(G)$ $(i+1)$-faces, and then $\bigcup_{i=1}^{r} C_i$ will have the desired face numbers. ∎

Acknowledgements. I would like to thank my thesis advisor Isabella Novik for suggesting this problem, and for her many useful discussions on solving the problem and writing an article.

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